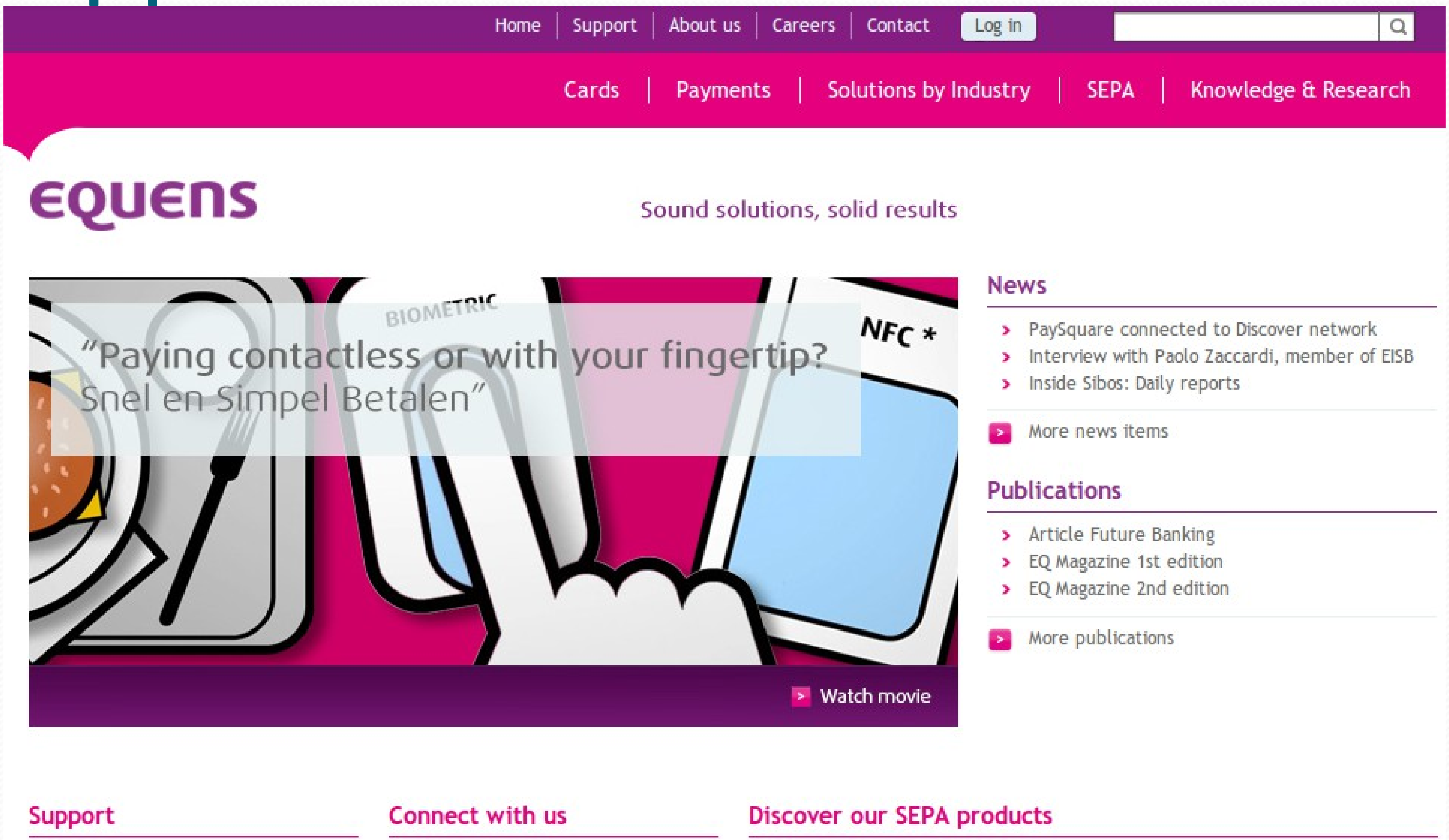
Fuzzy Sets and Fuzzy Logic

Application



Application

(From a press release)

Equens to offer RiskShield Fraud Protection for Card Payments

Today Equens, one of the largest pan-European card and payment Processors, announced that it has selected RiskShield from INFORM GmbH as the basis for a new approach to fraud detection and behaviour monitoring. By utilising the flexibility offered by RiskShield, Equens will be able to offer tailor-made fraud management services to issuers and acquirers.

UTRECHT, The Netherlands, 30/10/2012

Application

- From the brochure of “RiskShield”

RiskShield

is one of the world‘s leading software products for risk assessment and fraud prevention. RiskShield customers are typically banks, payment and processing service providers, telecommunication and insurance companies.

RiskShield

Application

ADVANTAGES OF

RiskShield

Based on Fuzzy Logic
Evaluates historical data such as typical user or contractual partner behaviour
Maximum performance (decisions within milliseconds)
High transparency for decisions
Short reaction times to new risk and fraud patterns
Reduces losses due to fraud or a missing risk assessment that can amount to millions

Crisp sets

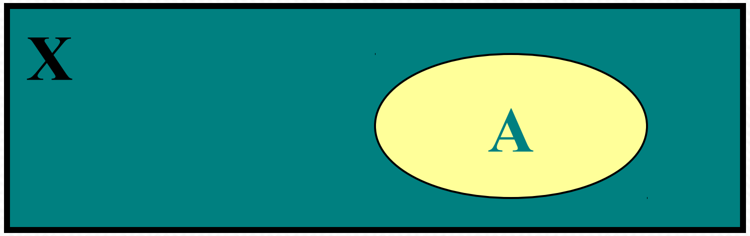
- Collection of definite, well-definable objects (elements).

Representation of sets:

- list of all elements

  $A=\{x_1, \ldots, x_n\}, x_j \in X$
- elements with property P

  $A=\{x | x \text{ satisfies } P\}, x \in X$
- Venn diagram



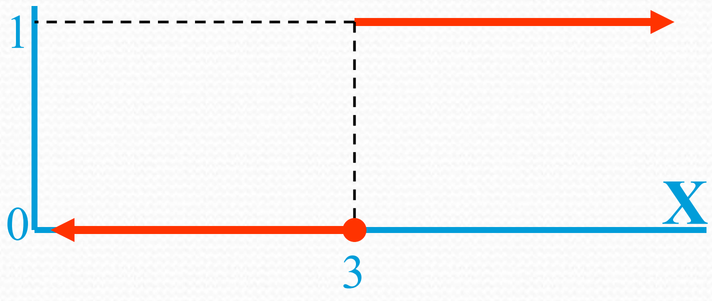
- characteristic function

  $f_A: X \rightarrow \{0,1\}$,

  $f_A(x) = 1, \Leftrightarrow x \in A$

  $f_A(x) = 0, \Leftrightarrow x \notin A$

Real numbers larger than 3:



Crisp (traditional) logic

- Crisp sets are used to define interpretations of first order logic

> If P is a unary predicate, and we have no functions, a possible interpretation is
> $A = \{0,1,2\}$
> $P^I = \{0,2\}$
> within this interpretation, $P(0)$ and $P(2)$ are true, and $P(1)$ is false.

- Crisp logic can be "fragile": changing the interpretation a little can change the truth value of a formula dramatically.

Fuzzy sets

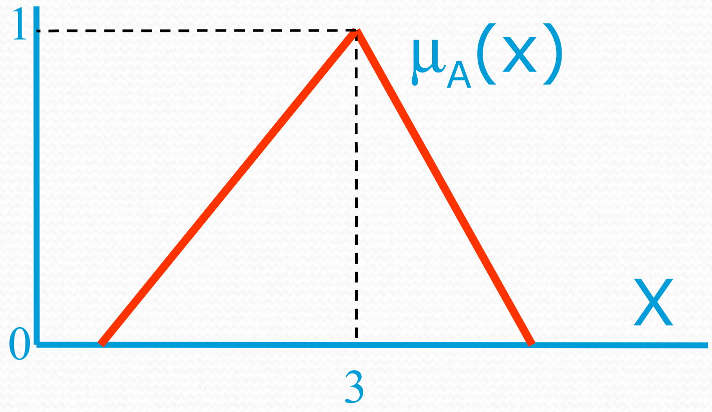
- Sets with fuzzy, gradual boundaries (Zadeh 1965)
- A fuzzy set A in X is characterized by its membership function $\mu_A: X \to [0,1]$

A fuzzy set A is completely determined by the set of ordered pairs

$A=\{(x,\mu_A(x)) | x \in X\}$

X is called the *domain* or *universe of discourse*

Real numbers about 3:



Fuzzy sets on discrete universes

- Fuzzy set C = "desirable city to live in"

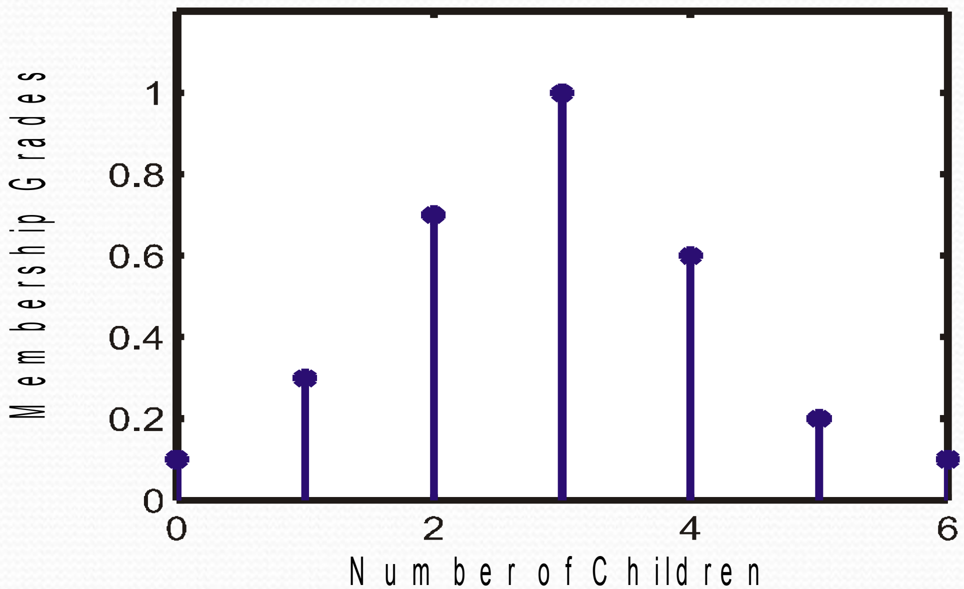
  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

- Fuzzy set A = "sensible number of children"

  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



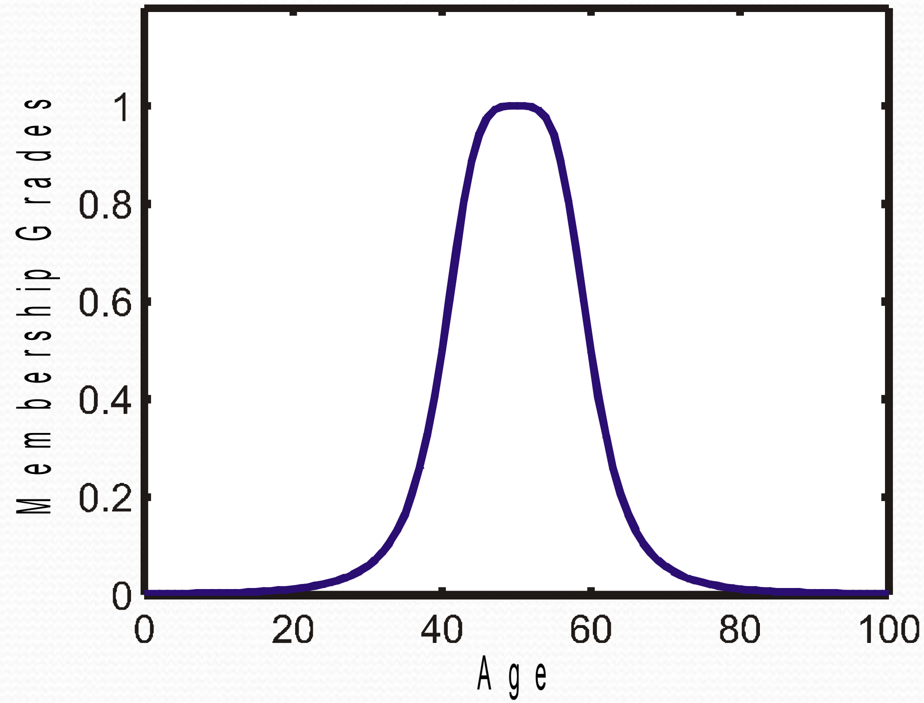
Fuzzy sets on continuous universes

- Fuzzy set B = “about 50 years old”

  X = Set of positive real numbers (continuous)

  B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



Membership Function formulation

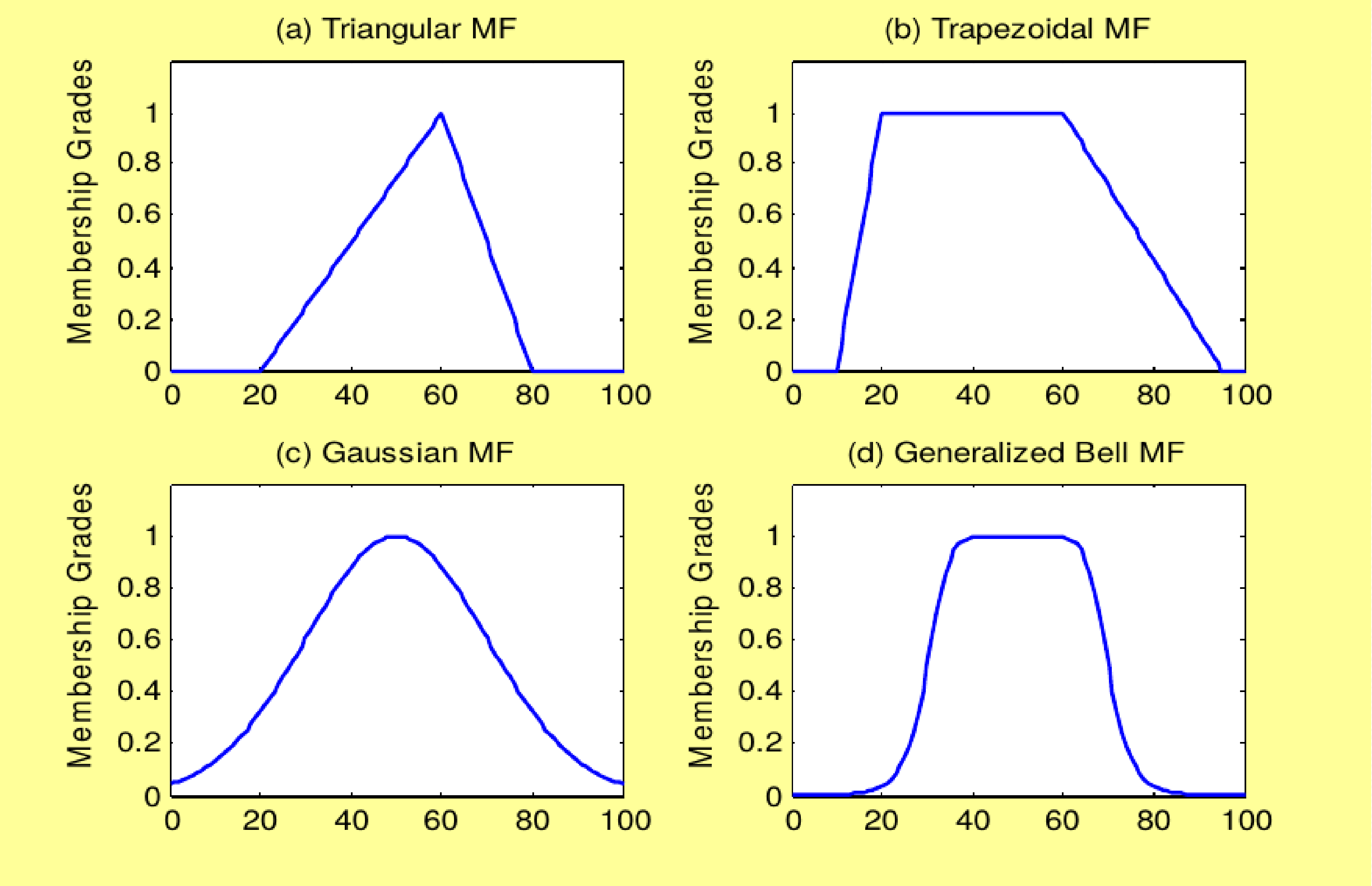
Triangular MF:
$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:
$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

Gaussian MF:
$$gaussmf(x;a,b) = e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$$

Generalized bell MF:
$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2a}}$$

MF formulation



Fuzzy sets & fuzzy logic

- Fuzzy sets can be used to define a level of truth of facts
- Fuzzy set C = "desirable city to live in"

  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

corresponds to a fuzzy interpretation in which
C(SF) is true with degree 0.9
C(Boston) is true with degree 0.8
C(LA) is true with degree 0.6
→ membership function $\mu_C(x)$ can be seen as a (fuzzy) predicate.

Notation

Many texts (especially older ones) do not use a consistent and clear notation

X is discrete

$$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

$$A = \sum_{x_i \in X} \mu_A(x_i)\ x_i$$

X is continuous

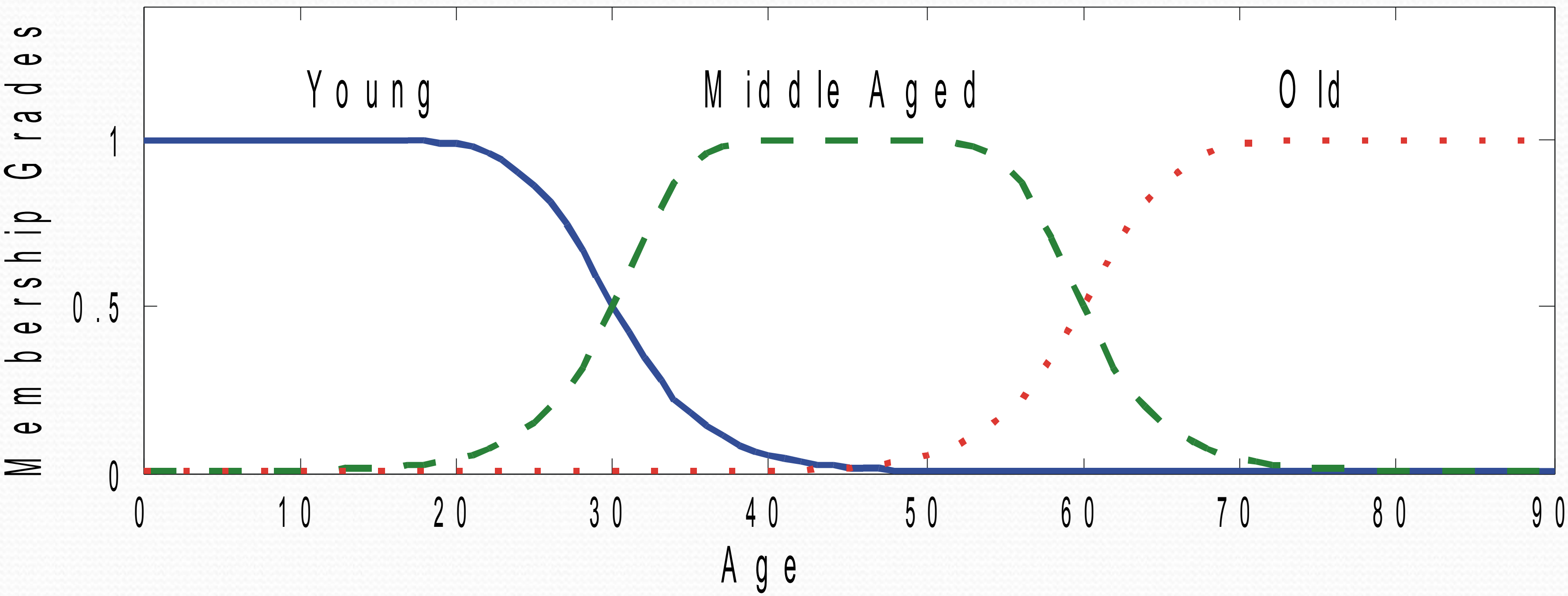
$$A = \int_X \mu_A(x) / x$$

$$A = \int_X \mu_A(x)\ x$$

Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

Fuzzy partition

Fuzzy partition formed by the linguistic values "young", "middle aged", and "old":



Fuzzy logic formulas

- Membership functions: $\mu_C(x), \mu_B(x)$
  - B="City is beautiful"
  - C="City is clean"
- Formulas: $\mu_C(x) \wedge \neg\mu_B(x), \mu_C(x) \vee \mu_B(x), \ldots$
- What is the truth value of such formulas for given *x*?
- We need to define a meaning for the connectives

Set theoretic operations /Fuzzy logic connectives

(Specific case)

- Subset:

$$A \subseteq B \Leftrightarrow \mu_A \leq \mu_B$$

- Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

- Union:

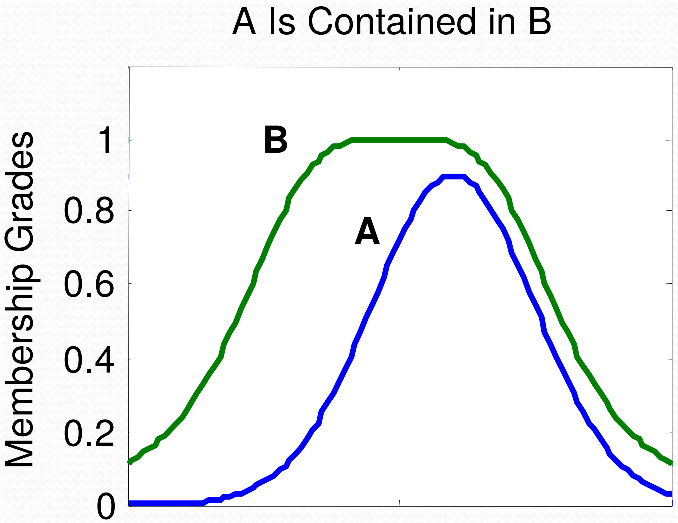
$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

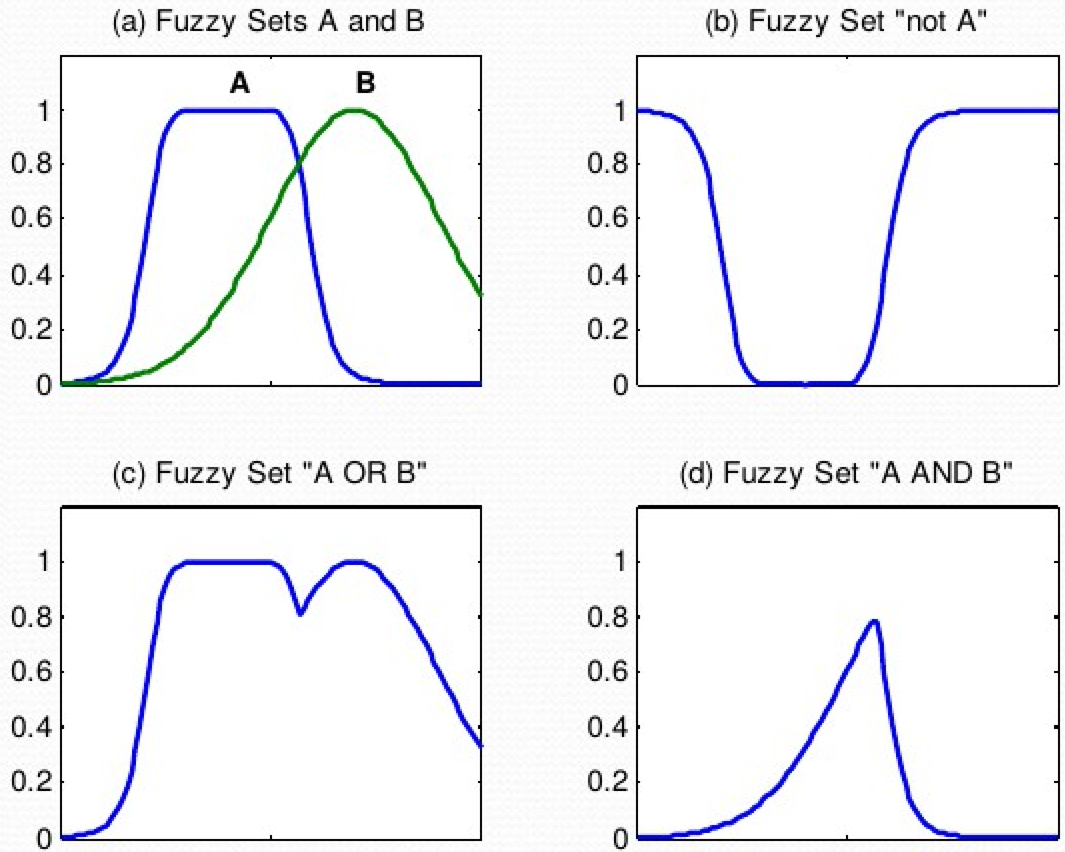
- Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

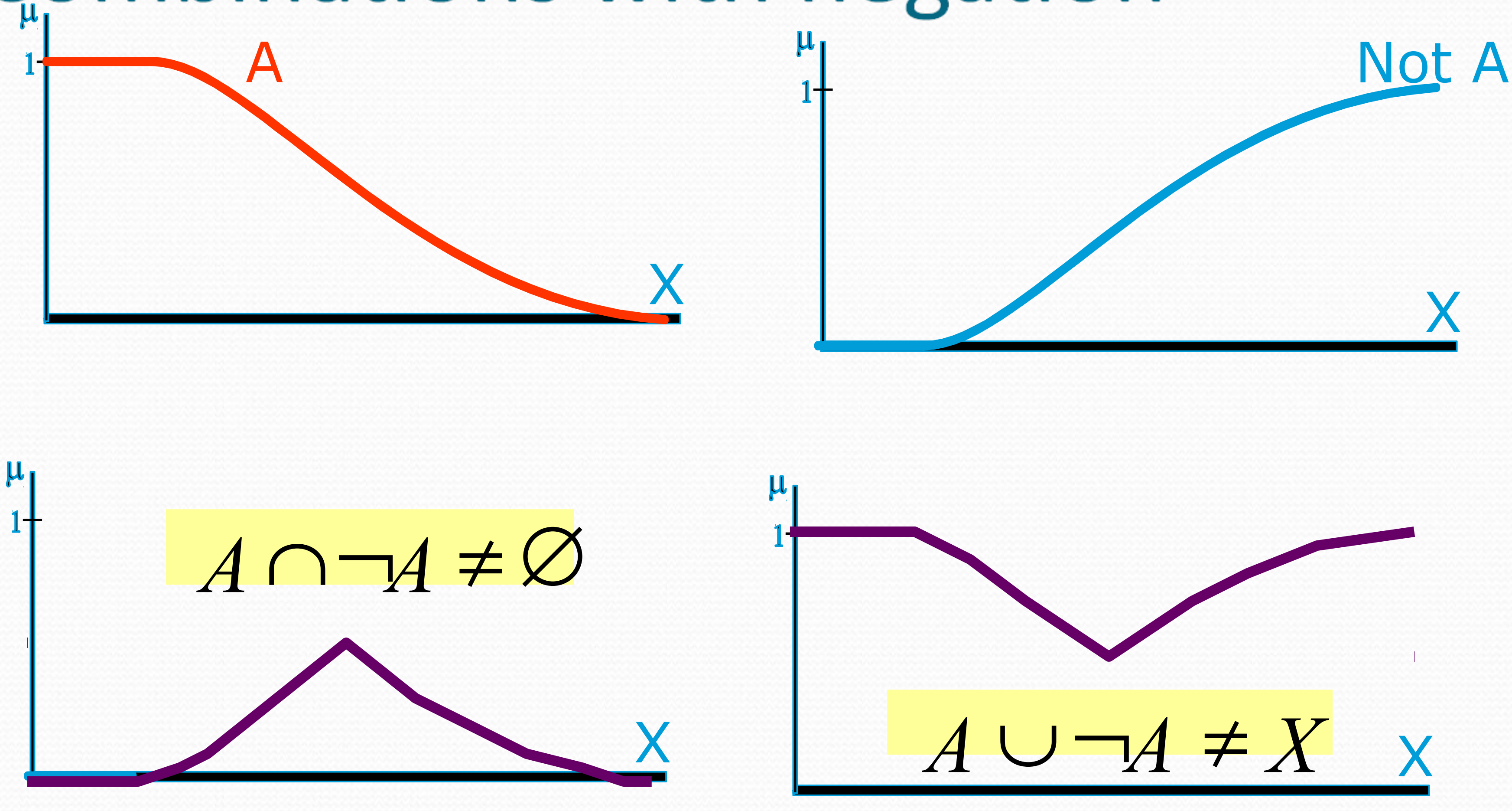
Set theoretic operations

$$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$$





Combinations with negation



Generalized negation

- General requirements:
  - Boundary: N(0)=1 and N(1) = 0
  - Monotonicity: N(a) > N(b) if a < b
  - Involution: N(N(a)) = a
- Two types of fuzzy complements:
  - Sugeno′s complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

  - Yager′s complement:

$$N_w(a) = (1-a^w)^{1/w}$$

Sugeno's and Yager's complements

Sugeno's complement:

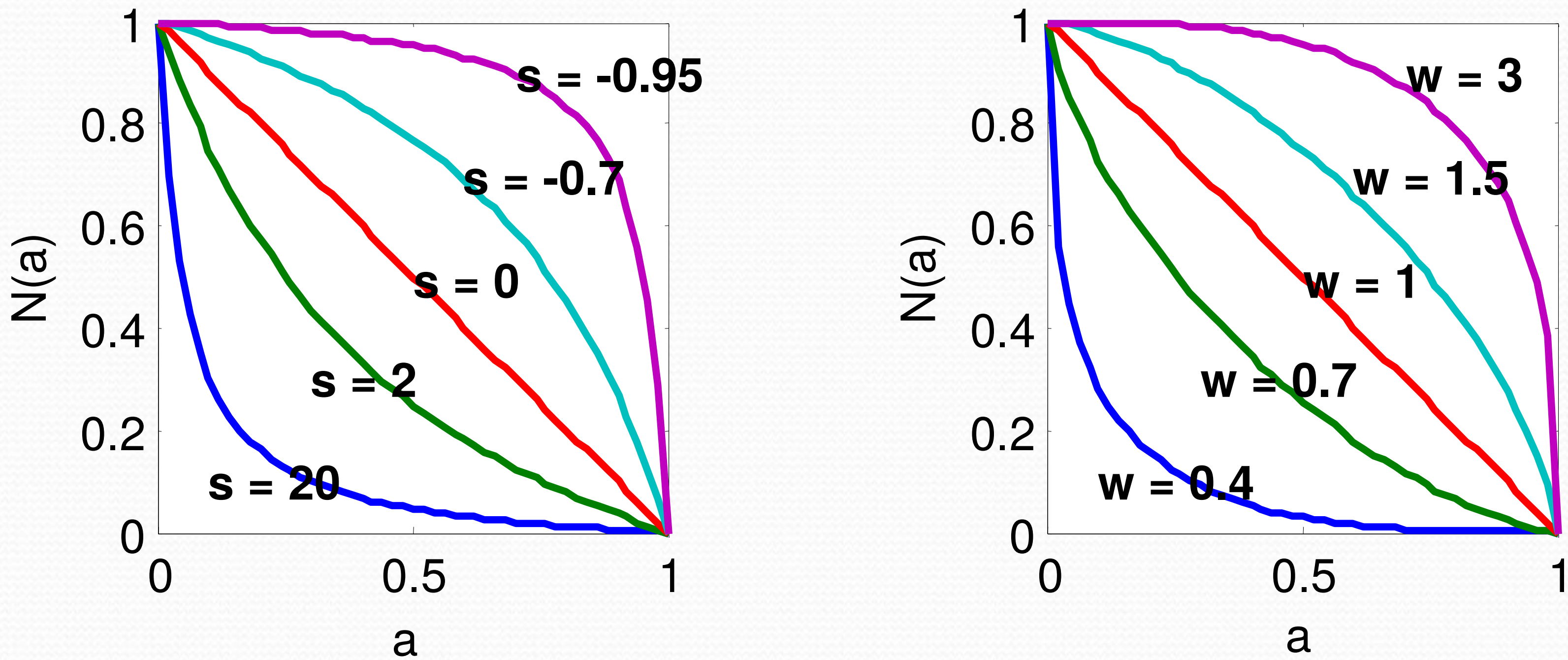
$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$

(a) Sugeno's Complements

(b) Yager's Complements



Generalized intersection (Triangular/T-norm, logical and)

- Basic requirements:
  - Boundary: $T(0, a) = T(a,0) = 0$, $T(a, 1) = T(1, a) = a$
  - Monotonicity: $T(a, b) <= T(c, d)$ if $a <= c$ and $b <= d$
  - Commutativity: $T(a, b) = T(b, a)$
  - Associativity: $T(a, T(b, c)) = T(T(a, b), c)$

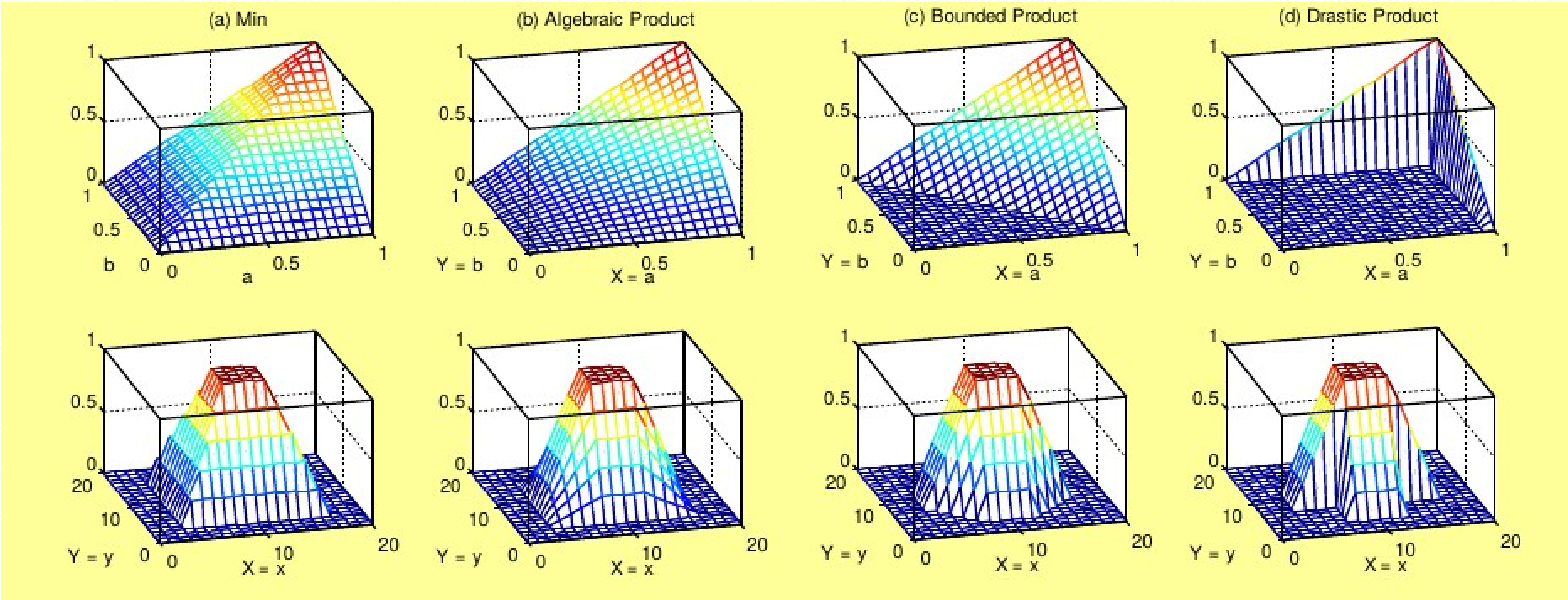
Generalized intersection (Triangular/T-norm)

- Examples:
  - Minimum: $T(a,b) = min(a,b)$
  - Algebraic product: $T(a,b) = a \cdot b$
  - Bounded product: $T(a,b) = max(0,(a+b-1))$
  - Drastic product:

$$T(a,b) = \begin{cases} a & if\ b=1 \\ b & if\ a=1 \\ 0 & otherwise \end{cases}$$

T-norm operator

Minimum: $T_m(a, b)$ ≥ **Algebraic product:** $T_a(a, b)$ ≥ **Bounded product:** $T_b(a, b)$ ≥ **Drastic product:** $T_d(a, b)$

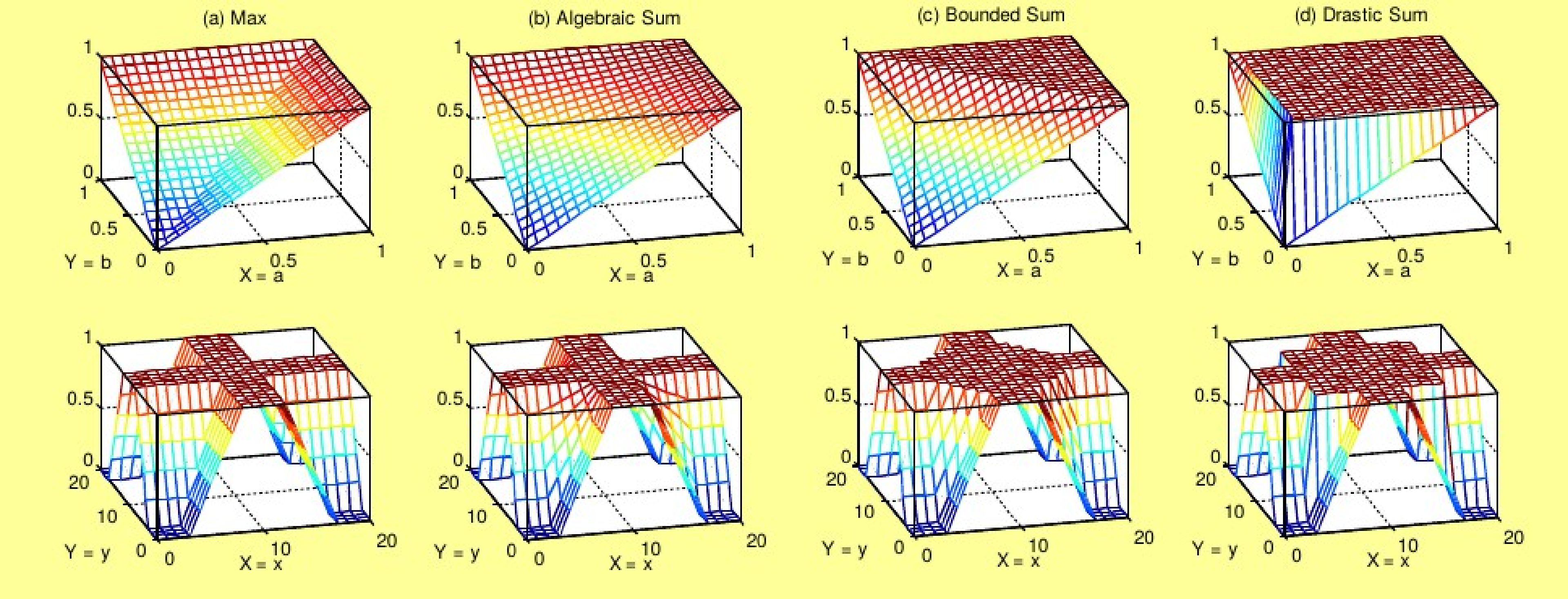


Generalized union (t-conorm)

- Basic requirements:
  - Boundary: S(1, a) = 1, S(a, 0) = S(0, a) = a
  - Monotonicity: S(a, b) < S(c, d) if a < c and b < d
  - Commutativity: S(a, b) = S(b, a)
  - Associativity: S(a, S(b, c)) = S(S(a, b), c)
- Examples:
  - Maximum: $S(a,b)=max(a,b)$
  - Algebraic sum: $S(a,b)=a+b-a\cdot b$
  - Bounded sum: $S(a,b)=min(1,(a+b))$
  - Drastic sum

T-conorm operator

Maximum: $S_m(a, b)$ $\leq$ **Algebraic sum: $S_a(a, b)$** $\leq$ **Bounded sum: $S_b(a, b)$** $\leq$ **Drastic sum: $S_d(a, b)$**



Generalized De Morgan's Law

- T-norms and T-conorms are duals which support the generalization of DeMorgan's law:
  - $T(a, b) = N(S(N(a), N(b)))$
  - $S(a, b) = N(T(N(a), N(b)))$